σ_0 = 10-18 kg/mm², were: n = 3, 59; A = $1.29 \cdot 10^{-7}$ $(kg/mm^2)^{-3.59} h^{-1}$; B = $2.21 \cdot 10^{-7}$ $(kg/mm^2)^{-3.59} h^{-1}$; r = 1.365. Theoretical curves of the creep (8), corresponding to the model (6), are plotted by the dash–dot lines in Fig. 5.

Table 2 gives values of the fracture time t* and the corresponding deformations p* with all the stresses considered. With g = 1, the above values of t* and p* were obtained in the experiments of [2, 3]. With g = 2, values of t* and p* corresponding to the last point of the curve (9) are given. With g = 3, the values of t* were calculated using (7), and the values of p* = A/B. It follows from the curves that each of the two theoretical models considered describes the experimental data of [2, 3] rather well.

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INVARIANT SOLUTIONS OF A THREE-DIMENSIONAL IDEAL PLASTICITY PROBLEM

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UDC 539.374

1. The state of the three-dimensional flow of an incompressible plastic medium can be described by using the following system of equations [1]:

$$p_{,1} = \frac{\sqrt{2}\, k_s u_{i,jj}}{2A} - \frac{\sqrt{2}\, k_s}{A^3} e_{ij} e_{mn} u_{mjn}, \tag{1}$$

$$u_{i,i} = 0, \quad A^2 = e_{ij} e_{ij}, \quad 2e_{ij} = u_{i,j} + u_{j,i} \ (i, j, m, n = 1, 2, 3),$$

where x_1, x_2, x_3 is a rectangular coordinate system, (u_1, u_2, u_3) is the velocity vector, p is the hydrostatic pressure, and k_s is the yield point. Summation is assumed to be over the repeated subscripts, and the subscript after the comma denotes differentiation with respect to the space variable with this subscript. Few exact solutions of this system are known at this time [2]. Axisymmetric solutions are not examined here, the papers [3-7] are devoted to them.

Let us use the method in [8] to seek particular solutions of the system (1.1). A group of continuous transformations allowed by the system (1.1) is generated by the following operators:

$$X_i = \partial/\partial x_i, \ Y_i = \partial/\partial u_i, \ M = x_i \partial/\partial x_i, \ N = u_i \partial/\partial u_i,$$
$$Z_1 = x_2 \partial/\partial x_3 - x_3 \partial/\partial x_2 + u_2 \partial/\partial u_3 - u_3 \partial/\partial u_2,$$
$$T_1 = x_2 \partial/\partial u_3 - x_3 \partial/\partial u_2, \ S = \partial/\partial p.$$

Four other operators Z_2, Z_3 and T_2, T_3 are obtained from Z_1, T_1 by circular commutation of the subscripts.

The group G_{15} is unsolvable, the operator S generates the center of this group. Let us construct optimal systems of the first, second, and third orders. They must be constructed in order to seek substantially different solutions in the group sense. Let us mention certain invariant solutions.

2. The invariant solution in the subgroup $\langle X_3 + T_1 + \alpha T_2 + \beta T_3 \rangle$ was found in [9]. It describes the flow of a prismatic rod of plastic material that is subjected to tension, torsion, and bending simultaneously.

The solution in the subgroup $\langle X_1 + T_1 + \alpha T_2, \ X_2 - T_2 + \beta T_1 \rangle$ was investigated in [10]. This solution describes the kinetic field corresponding to a homogeneous state of stress.

The invariant solution relative to the subgroup $\langle T_1 + X_1, \ \alpha X_1 + X_2 - T_2 + \beta Y_1 + \gamma Y_2 \rangle$ was studied in [11], where it is interesting in that it depends on 17 constants.

Krasnoyarsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 159-163, May-June, 1980. Original article submitted March 12, 1979.

0021-8944/80/2103-0417$07.50

The solution in the subgroup $\langle X_3 + Y_3 \rangle$ was examined in [12]. It has the form

$$u_1 = ax_1,\ u_2 = bx_2,\ u_3 = -(a+b)x_3 + f(x_1, x_2). \tag{2.1}$$

Here

$$\begin{gathered} e_{11} = a,\ e_{13} = f_{,1},\ e_{23} = f_{,2}, \\ e_{22} = b,\ e_{12} = 0,\ e_{33} = -(a+b), \\ \sigma_{11} = -p(x_1, x_2) + \lambda a,\ \sigma_{22} = -p(x_1, x_2) + \lambda b, \\ \sigma_{33} = -p(x_1, x_2) - (a+b)\lambda,\ \sigma_{12} = 0, \\ \sigma_{13} = f_{,1}\lambda,\ \sigma_{23} = f_{,2}\lambda,\ \lambda = \sqrt{2}\, k_s (e_{ij}e_{ij})^{-1/2}, \end{gathered}$$

and the function $p(x_1, x_2)$ is determined from the equilibrium equations.

If $a \neq b$, we then obtain an extension of the known Prandtl solution for a plastic mass compressed between two slabs to the three-dimensional case.

If $a = b$, then just one of the possible solutions is obtained in [12]: $f = \sqrt{1 - 4(x_1 - p)^2 - 4(x_2 - q)^2}$. Let us find the other solution in this case. Substituting (2.1) into (1.1), we have

$$\begin{gathered} p = a\lambda + c_1, \\ \left(\frac{f_{,1}}{\sqrt{6a^2 + f_{,1}^2 + f_{,2}^2}}\right)_{,1} + \left(\frac{f_{,2}}{\sqrt{6a^2 + f_{,1}^2 + f_{,2}^2}}\right)_{,2} = c_2, \end{gathered} \tag{2.2}$$

where c_1, c_2 are arbitrary constants.

For $c_2 = 0$ Eq. (2.2) goes over into the well-studied equation of minimal surface. In particular, f can be taken in the form

$$1)\ f = \sqrt{6}\, a \ln\frac{\cos x_2}{\cos x_1},\quad 2)\ f = \sqrt{6}\, a \operatorname{arctg}\frac{x_2}{x_1},\quad 3)\ f = \sqrt{6}\, a \operatorname{Arch}\sqrt{x_1^2 + x_2^2}.$$

The appropriate state of stress is determined from (2.2).

3. Let us seek the solution in the subgroup $\langle X_3 + \varepsilon N \rangle$. It has the form

$$u_i = u_i^*(x_1, x_2) \exp \varepsilon x_3, \quad p = p(x_1, x_2). \tag{3.1}$$

Substituting (3.1) into (1.1), we obtain a system S/H in the functions u_i^*, p

$$\begin{gathered} (\lambda u_{1,1}^*)_{,1} + \frac{1}{2}[\lambda(u_{1,2}^* + u_{2,1}^*)]_{,2} = p_{,1}, \\ \frac{1}{2}[\lambda(u_{1,2}^* + u_{2,1}^*)]_{,1} + (\lambda u_{2,2}^*)_{,2} = p_{,2}, \\ [\lambda(u_{3,1}^* + \varepsilon u_1^*)]_{,1} + [\lambda(u_{3,2}^* + \varepsilon u_2^*)]_{,2} = 0, \\ u_{1,1}^* + u_{2,2}^* + \varepsilon u_3^* = 0,\quad \lambda = \sqrt{2}\, k_s[(u_{1,1}^*)^2 + (u_{2,2}^*)^2 + \\ + (u_{1,2}^* + u_{2,1}^*)^2 + (u_{3,1}^* + \varepsilon u_1^*)^2 + (u_{3,2}^* + \varepsilon u_2^*)^2]^{-1/2}. \end{gathered} \tag{3.2}$$

Assuming ϵ infinitesimal, we expand the desired functions in a series in this parameter

$$u_i^* = \sum_{k=0}^{\infty} \varepsilon^k u_i^{(k)}(x_1, x_2), \quad p = \sum_{k=0}^{\infty} \varepsilon^k p^{(k)}(x_1, x_2).$$

If $u_3^{(0)}(x_1, x_2) \equiv 0$ then $u_1^{(0)}(x_1, x_2), u_2^{(0)}(x_1, x_2), p^{(0)}(x_1, x_2)$ are the solution of the plane problem of ideal plasticity. In this case, we have from (3.2) for the i-th approximation

$$\begin{gathered} (\lambda u_{1,1}^{(i)})_{,1} + \frac{1}{2}[\lambda(u_{1,2}^{(i)} + u_{2,1}^{(i)})]_{,2} = p_{,1}^{(i)}, \\ \frac{1}{2}[\lambda(u_{1,2}^{(i)} + u_{2,1}^{(i)})]_{,1} + (\lambda u_{2,2}^{(i)})_{,2} = p_{,2}^{(i)}, \\ u_{1,1}^{(i)} + u_{2,2}^{(i)} + u_3^{(i-1)} = 0, \\ [\lambda(u_{3,1}^{(i)} + u_1^{(i-1)})]_{,1} + [\lambda(u_{3,2}^{(i)} + u_2^{(i-1)})]_{,2} = 0. \end{gathered} \tag{3.3}$$

Here

$$\lambda = \sqrt{2}\, k_s[(u_{1,1}^{(0)})^2 + (u_{2,2}^{(0)})^2 + (u_{1,2}^{(0)} + u_{2,1}^{(0)})^2]^{-1/2}$$

After having determined all $u_i^{(k)}$ from (3.1), we have

$$u_i = \sum_{k=0}^{\infty} \varepsilon^k \left(\sum_{l=0}^{k} u_i^{(l)} x_3^{k-l} \right).$$

Let us note that the first-approximation velocity field

$$u_i = u_i^{(0)} + \varepsilon u_i^{(1)} x_3$$

describes a stress–strain state of a beam in the plane strain plastic state and subjected to the action of an infinitesimal torque

$$G_z = \varepsilon \int\int \left(\sigma_{32}^{(1)} x_1 + \sigma_{31}^{(1)} x_2 \right) dx_1 dx_2.$$

Now, let $u_1^{(0)} = u_2^{(0)} = 0$, and $u_3^{(0)} = \text{const}$, $p^{(0)} = \text{const}$.

In the zeroth approximation, (1.1) is satisfied identically.

In the i-th approximation we have the system of equations (3.3), but here

$$\lambda = \sqrt{2}\, k_s \left| u_3^{(0)} \right|^{-1} = \text{const}.$$

This velocity field can be interpreted as follows. In the half-space $x_3 \leqslant 0$ let a thin edge be impressed. Its equation is

$$\varepsilon x_3 = f(x_1, x_2). \tag{3.4}$$

In this case the zeroth approximation will correspond to the impression of a zero-thickness edge.

The normal to the surface of the edge is written in the form

$$\mathbf{n} = \frac{\partial f}{\partial x_1} \mathbf{i} + \frac{\partial f}{\partial x_2} \mathbf{j} - \varepsilon \mathbf{k}.$$

In a first approximation the velocity vector has the form

$$\mathbf{V} = \varepsilon u_1^{(1)} \mathbf{i} + \varepsilon u_2^{(1)} \mathbf{j} + \left(u_3^{(0)} + \varepsilon u_3^{(1)} \right) \mathbf{k}.$$

The velocity vector on the edge surface lies in a plane tangent to the edge, consequently

$$(\mathbf{V}, \mathbf{n}) = \varepsilon u_1^{(1)} \frac{\partial f}{\partial x_1} + \varepsilon u_2^{(1)} \frac{\partial f}{\partial x_2} - \varepsilon \left(u_3^{(0)} + \varepsilon u_3^{(1)} \right). \tag{3.5}$$

Linearizing (3.5), we obtain

$$u_3^{(0)} = u_1^{(1)} \frac{\partial f}{\partial x_1} + u_2^{(1)} \frac{\partial f}{\partial x_2} \quad \text{for} \quad x_3 = 0. \tag{3.6}$$

To determine $u_1^{(1)}$, $u_2^{(1)}$ it is now necessary to solve (3.3) with the boundary conditions (3.6). We obtain the boundary condition analogously in the successive approximations and we take account of the fact that $u_3^{(i)} = 0$ (i = 1, 2, 3, ...) on the edge surface.

Let us note that the case when a plane edge is impressed, i.e., x_2 is not in the right side of (3.4), was studied in [4].

4. We seek the invariant solution in the subgroup $\langle X_2 - X_3,\ X_3 + N + \alpha S \rangle$ in the form

$$\begin{gathered} u_1 = u(x_1) \exp \xi,\ u_2 = v(x_1) \exp \xi,\ u_3 = w(x_1) \exp \xi, \\ p = p(x_1) + a\xi,\ \xi = x_2 + x_3,\ \alpha a = 1. \end{gathered} \tag{4.1}$$

Taking (4.1) into account, the system in the stress deviator components is written in the form

$$\begin{gathered} \frac{\partial S_1}{\partial x_1} = \frac{\partial p}{\partial x_1},\quad \frac{\partial \tau_{12}}{\partial x_1} = \frac{\partial \tau_{13}}{\partial x_1} = Q, \\ S_1^2 + S_2^2 + S_3^2 + 2\left(\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2 \right) = 2k_s^2, \\ S_1 + S_2 + S_3 = 0,\ S_1 = \lambda u',\ S_2 = \lambda v,\ S_3 = \lambda w, \\ 2\tau_{12} = \lambda(u + v'),\ 2\tau_{13} = \lambda(v + w'), \\ 2\tau_{23} = \lambda(v + w),\ \lambda = \sqrt{2} k_s (e_{ij} e_{ij})^{-1/2}. \end{gathered} \tag{4.2}$$

The prime denotes differentiation with respect to x_1. From (4.2) we have

$$\tau_{12} = a x_1 + c_1,\ \tau_{13} = Q x_1 + c_2.$$

If $c_1 = c_2$, then $v' = w'$. Let us consider $v = w$. Then

$$\tau_{23} = 2S_2,\ S_1 = 2S_2,\ S_2 = S_3.$$

We have from the fluidity condition

$$S_2 = \pm \frac{\sqrt{7}}{7} \sqrt{k_s^2 - 2\tau_{12}^2}.$$

The stress tensor components are written in the form

$$\sigma_1 = c = \text{const},\ \sigma_2 = 3S_2 + c,\ \sigma_3 = 3S_2 + c,$$
$$\tau_{23} = 2S_2,\ \tau_{12} = \tau_{13} = ax_1 + c_1.$$

This solution can be interpreted as the three-dimensional flow of a plastic material between slabs parallel to the Ox_2x_3 plane, which approach each other along the Ox_1 axis.

The velocity field is determined from the solution of the system

$$u' + 2v = 0,\quad u'/(u + v') = \frac{\sigma_1}{2\tau_{12}}$$

with the boundary conditions

$$u(l) = -V,\ u(-l) = V,\ v(l) = v(-l) = 0,$$

where V is the slab velocity along the Ox_1 axis, and $2l$ is the distance between the slabs.

The author is grateful to B. D. Annin for formulating the problem and discussing the results.

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